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Mamadu Decomposition Transform Method for Time-Space Fractional Telegraph Equation with Mittag-Leffler function

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Abstract: In this paper, the Mamadu Decomposition Transform Method (MDTM) is developed and implemented for the solution of the time-space fractional telegraph equation using the Mittag-Leffler function. The MDTM is a hybrid analytic-numerical technique designed to resolve complex differential equations, requiring no form of domain discretization, thereby reducing computational cost and complexity. The MDTM integrates the Mamadu Transform and Adomian Decomposition Method (ADM) to obtain reliable, accurate, and efficient solutions. The method (MDTM) is experimented on with the proposed problem by varying the parameters α, β, and μ. Resulting numerical evidence obtained from MDTM is compared with the analytic solution expressed using the Mittag-Leffler function and other methods in literature, such as the Adomian Decomposition Method (ADM), Homotopy Analysis Method (HAM), Homotopy Perturbation Method (HPM), Finite Difference Method (FDM), Fractional Differential Transform Method (FDTM), and Spectral Tau Method, for demonstrating effectiveness, accuracy, and excellent convergence. The finding reflects the effectiveness and efficiency of MDTM in resolving fractional order telegraph equations, making it a viable tool for resolving other Partial Differential Equations (PDEs).

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I. INTRODUCTION

The concept and foundation of fractional calculus can be traced back to 1695 with the first mention of half-differentiation in the literature. However, fractional calculus did not gain substantial traction until the development of an increasing number of real-world systems that exhibit the ability to store memory and have a history. These types of behaviours occur in a variety of systems that are typically well represented by integer-order models; however, fractional-order models better represent many of these phenomena because they can capture non-local dynamics that occur in various physical and engineering phenomena. Due to this ability, fractional calculus is now used in areas such as quantum mechanics, viscoelastic and rheological systems, fractional kinetics, propagation of electromagnetic waves, dielectric relaxation, flow and network dynamics, interactions between the electrode and the electrolyte, and in biological, biomedical, and ecological systems [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

In two decades, there has been much progress researching fractional differential equations (FDEs) through development of numerical and analytical methods for solving a variety of fractional-order systems. Examples of such studies include diffusion models combined with Buckmaster's fractional model [12], the time–space fractional telegraph equation [13], the fractional Korteweg–de Vries–Kuramoto-Burger (KdV-KB) equation [14], the fractional flow and traffic systems [15], the fractional Drinfeld-Sokolov-Wilson model [16], the fractional anomalous and subdiffusion equations [17], the fractal Burgers' equations [18], and the fractional Navier-Stokes system [19]. Also, researchers created several epidemiological models describing the dynamics of pine wilt and tuberculosis using fractional orders [20, 21]. These results suggest that fractional-order equations can capture the same level of complexity as their integer-order counterparts do in many instances within engineering, physics, and biology.

The telegraph equation has received a considerable amount of attention due to its purpose of modelling the transmission of signals and the propagation of waves through a medium. The classical telegraph equation is used as a model for many things, such as electrical transmission lines, vibration, and signal processing. The fractional telegraph equation (i.e., time-space fractional telegraph equation) introduces derivatives of non-integer order with respect to time and/or space. The fractional telegraph equation models the propagation of wave and inherent memory in the diffusion phenomenon. [22] The fractional telegraph equation has been employed in precisely modelling the response of viscoelastic material; electromagnetic media with memory; and biological responses for transmitting/ receiving signals from the biological systems. For example, the fractional telegraph equation has been employed to model blood flow travelling through arterial systems and charged particles travelling through transmission lines, and in each case provided more accurate modelling than equivalent integer-order models [23].

Although many researchers have investigated fractional telegraph equations, few analytical solutions have been found for general cases and most of the solutions found had to rely on some form of simplifying assumption. Many different techniques have been used to obtain an analytic solution for more general cases of the fractional telegraph model including the Fourier and Laplace transform methods [24], the Homotopy Analysis Method (HAM) [25], the Sumudu transform [26], and the Adomian Decomposition Method (ADM) [27, 28, 29, 30, 31]. While these techniques have provided some important information about the general fractional telegraph model, all of them suffer from limitations with respect to slow convergence, difficulties dealing with nonlinearities, and/or extreme complexity associated with non-local fractional operators when combined with initial and boundary conditions.

Integral transforms are a robust method for solving differential and integral equations. Classic integral transforms (Laplace, Fourier, Mellin, and Sumudu) have been used numerous times to work on FDEs. Recently, newer forms of integral transforms (Elzaki, Sawi, Shehu, Mohand, and Mahgoub) that yield analytical and numerical solutions have emerged and are showing great promise as technique [32, 33, 34, 35, 36, 37, 38]. There is, however, little attention to integral transforms designed specifically for fractional telegraph equations when combined with decomposition techniques that allow for recursive solution development.

In an effort to bridge that gap, the current work uses the recently created Mamadu Transform (MT) to create a new hybrid analytic/numeric method to solve fractional telegraph equations. The Mamadu Transform has demonstrated success as a means of treating certain types of complex differential and integral equations; however, it has yet to be investigated fully for use with the fractional telegraph equation modeL[39, 40, 41, 42, 43, 44].

This study offers a new hybrid approach, called the Mamadu Decomposition Transform Method (MDTM), which combines both the Mamadu Transform and the Adomian Decomposition Method (ADM), for solving the time-space fractional telegraph equation as defined in terms of the Mittag-Leffler function. In comparison to existing methods that exclusively utilize either transforms or decomposition schemes separately, this combined approach will provide improved accuracy, increased speed of convergence, and have more applications for nonlinear and non-local fractional systems.

The aim of this research is to create a reliable and computationally efficient analytical series solution for the time-space fractional telegraph equation, as represented by the Multi-Dimensional Transform Method (MDTM). This article represents the first time that the Mamadu Transform has been utilized with the Adomian Decomposition Method (ADM) in order to formulate an analytical series solution for fractional telegraph equations.

Summarization of the contributions made through this study is as follows:

1. MDTM combines the advantages of the Mamadu Transform (MT) and Adomian Decomposition Method (ADM) to create a Hybrid Analytic/Numerical approach that can be used to solve fractional differential equations (FDEs).

2. The MDTM extends the MT by coupling it with ADM, allowing for an extension of existing research concerning the use of individual Integral Transformations to solve Fractional Models to now include a continual cofactor of both integral transformation techniques for the solution of Time/Space Fractional Telegraph Equations (FTTE).

3. The MDTM exhibits increased accuracy of convergence and accuracy of solutions as compared to most method of analytical or semi-analytical approach's applied to problems containing nonlocal operator and/or complex Initial/Boundary Conditions (IBC's).

II. PRELIMINARIES

A. Definition

Mamadu Transform (M.T) was developed by Mamadu in 2025 [39, 43, 44]. The transform is defined for a class of functions $u(t)$ with exponential order in a set A given as

$$A = \{u(t) : \exists a < s \text{ such that } |u(t)| \leq Ce^{at} \text{ for some constant } C, \\ \text{and } u(t) \text{ is continuous in } [0,\infty)\}. \tag{2.1}$$

The M.T is defined by

$$M\{u(t)\}(s) = \int_0^\infty (1+t^2)e^{-st}u(t)dt,\ s > 0. \tag{2.2}$$

The inverse of M.T is given as

$$u(t) = M^{-1}\{M(u(t))(s)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} M[u](s).(1+t^2)e^{-st}dt, \tag{2.3}$$

where γ is a constant to aid contour integration and convergence. The advantages of Mamadu Transform over classical transforms can be found in the reference [39].

B. Definition

M.T for the function $u(t)$ with fractional order α in the Caputo sense is generalized by [39, 43, 44]

$$M\left[\frac{\partial^\alpha u(t)}{\partial t^\alpha}\right](s) = s^\alpha M[u(x,t)](s) \\ -\sum_{j=0}^{[\alpha]-1}\left(s^{\alpha-1-j} + \gamma_j(s)s^{\gamma-2-j} + \beta_j(s)s^{\gamma-2-j}\right)u^{(j)}(x,0). \tag{2.4}$$

where α is the Caputo fractional derivative, while $\gamma_j(s)$ and $\beta_j(s)$ are coefficient functions arising from the expansion of the kernel associated with the weight function $(1+t^2)$.

C. Definition

Caputo Fractional derivative with order α is defined by [1, 2, 3]

$$D_t^\alpha u(t) = \begin{cases} \frac{\partial^n u(t)}{\partial t^n}, & \alpha = n \in \mathbf{N}, \\ \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}u^{(n)}(s)ds, & n-1 < \alpha < n \end{cases} \tag{2.5}$$

D. Definition

Mittag-Leffler, $E_\alpha(x)$ for $\alpha > 0$ is defined as [4, 5]

$$E_\alpha(x) = \sum_{n=0}^\infty \frac{x^n}{\Gamma(n\alpha+1)}, \quad \alpha > 0, \quad x \in \mathbf{C}. \tag{2.6}$$

III. MAIN RESULTS

We consider in this section a time–space fractional telegraph equation of the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+\frac{\partial^\beta u(x,t)}{\partial x^\beta}+\mu u(x,t)=0, \quad (x,t)\in(0,L)\times(0,T). \tag{3.1}$$

with initial and boundary conditions

$$u(x,0)=f(x), \quad \left.\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right|_{t=0}=g(x), \quad x\in[0,L] \tag{3.2}$$

$$u(0,t)=0, \quad u(L,t)=0, \quad 0<t<T, \tag{3.3}$$

where T is time domain, and L is the length of the spatial domain, $u(x,t)$ is the unknown function, $\alpha\in(0,1]$ is the order of the Caputo fractional derivative, $\beta\in(0,2)$ is the order of the Riesz space–fractional derivative, μ is a constant coefficient, $f(x)$ represents the initial spatial distribution of $u(x,t)$ and $g(x)$ is the initial condition for the time fractional derivative.

Taking the Mamadu transform to both sides of the equation (3.1), we have

$$M\left[\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right](s)+M\left[\frac{\partial^\beta u(x,t)}{\partial x^\beta}\right](s)+\mu M[u(x,t)](s)=0. \tag{3.4}$$

By definition,

$$M\left[\frac{\partial^\alpha u(t)}{\partial t^\alpha}\right](s)=s^\alpha M[u(x,t)](s)-\sum_{j=0}^{[\alpha]-1}\left(s^{\alpha-1-j}+\gamma_j(s)^{\alpha-2-j}+\beta_j(s)^{\alpha-3-j}\right)\times u^{(j)}(x,0), \tag{3.5}$$

where $j-1<\alpha\leq j$. If the weight $(1+t^2)$ produces no extra boundary contributions, then $\gamma_j(s)\equiv 0$ and $\beta_j(s)\equiv 0$ for all j, and (3.5) reduces to the familiar Caputo transform identity for $\alpha\leq 1$,

$$M\left[\frac{\partial^\alpha u}{\partial t^\alpha}\right]=s^\alpha M[u(x,t)](s)-s^{\alpha-1}u(x,0). \tag{3.6}$$

Substituting (3.6) into (3.4), we have

$$\begin{aligned}&s^\alpha M[u(x,t)](s)-s^{\alpha-1}u(x,0)+M\left[\frac{\partial^\beta u(x,t)}{\partial x^\beta}\right](s)+\mu M[u(x,t)](s)=0\\ \Rightarrow\ &M\left[\frac{\partial^\beta u(x,t)}{\partial x^\beta}\right](s)+(s^\alpha+\mu)M[u(x,t)](s)=s^{\alpha-1}f(x).\end{aligned}$$

For the space–fractional derivative, we have that

$$M\left[\frac{\partial^\beta u(x,t)}{\partial x^\beta}\right]=\frac{\partial^\beta M[u(x,t)]}{\partial x^\beta}.$$

Thus,

$$\frac{\partial^\beta M[u(x,t)]}{\partial x^\beta}+(s^\alpha+\mu)M[u(x,t)](s)=s^{\alpha-1}f(x), \tag{3.7}$$

which is a space–fractional differential equation, which can be solved using the eigen function expansion technique or method of separation of variables.

Let $\lambda=s^\alpha+\mu$, and the Mamadu transform parameter satisfies $s>0$, we have $s^\alpha>0, s^\alpha>0$. Hence, $\lambda>0$ whenever $\mu\geq 0$. Therefore, (3.7) becomes

$$\frac{\partial^\beta M[u(x,t)]}{\partial x^\beta}+\lambda M[u(x,t)]=0. \tag{3.8}$$

The complementary solution of (3.8) for $\beta=2$, the solution is classical given as

$$M[u](x,t)=C_1e^{-\sqrt{\lambda}x}+C_2e^{\sqrt{\lambda}x}, \quad C_1 \text{ and } C_2 \text{ are a constant.} \tag{3.9}$$

Applying boundary conditions on (3.9), we obtain a hyperbolic sine solutions given as

$$M[u](x,t)=C\sinh\left(\sqrt{\lambda}(L-x)\right), \quad \text{since } C_1=-C_2.$$

For general β, let assume a power series solution to (3.8) as

$$M[u](x,t)=\sum_{n=0}^{\infty}a_nx^{\beta n}.$$

Using the known fractional identity

$$\frac{\partial^\beta M[u(x,t)]}{\partial x^\beta}=\frac{\Gamma(\beta n+1)}{\Gamma(\beta n+1-\beta)}x^{\beta n-\beta},$$

on (3.8) yields the recurrence relation,

$$a_{n+1}=-\frac{\lambda}{\Gamma(\beta(n+1)+1)}a_n\Gamma(\beta n+1).$$

Solving the above relation yields,

$$a_n=\frac{(-\lambda)^n}{\Gamma(\beta n+1)}a_0.$$

Hence, using the Mittag-Leffler eigenfunctions, we have,

$$\begin{aligned}M[u](x,t)&=a_0\sum_{n=0}^{\infty}\frac{(-\lambda x^\beta)^n}{\Gamma(\beta n+1)}\\&=CE_\beta(-\lambda x^\beta),\end{aligned}$$

where $a_0=C$.

Let the particular solution be assumed as

$$M[u_p](x,t)=A(x),$$

where $A(x)$ is a function to be computed. Thus, (3.8) becomes

$$\frac{\partial^\beta A(x)}{\partial x^\beta} + \lambda A(x) = s^{\alpha-1} f(x)$$
$$\Rightarrow \quad A(x) = \frac{s^{\alpha-1} f(x)}{\lambda} = \frac{s^{\alpha-1} f(x)}{\lambda},$$

since $A(x)$ is a constant with respect to x. This implies that

$$M[u_p](x,t) = \frac{s^{\alpha-1} f(x)}{\lambda}. \tag{3.10}$$

Thus, the general solution is given by

$$M[u(x,t)](s) = CE_\beta(-\lambda x^\beta) + \frac{s^{\alpha-1} f(x)}{\lambda}, \tag{3.11}$$

C is determined using the boundary conditions.

To find $u(x,t)$ in (3.11), we apply the inverse Mamadu transform [39, 43, 44] given by

$$u(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\alpha}^{\gamma+i\alpha} M[u(x,t)](s)(1+t^2)e^{-st}ds.$$

From (3.11),

$$u(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\alpha}^{\gamma+i\alpha} \left[CE_\beta(-\lambda x^\beta) + \frac{s^{\alpha-1} f(x)}{\lambda}\right] \times (1+t^2)e^{-st}ds.$$

$$u(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\alpha}^{\gamma+i\alpha} CE_\beta(-\lambda x^\beta)(1+t^2)e^{-st}ds + \frac{1}{2\pi i}\int_{\gamma-i\alpha}^{\gamma+i\alpha} \frac{s^{\alpha-1} f(x)}{\lambda}(1+t^2)e^{-st}ds.$$

We evaluate the two terms separately as follows:

First term:

$$\frac{1}{2\pi i}\int_{\gamma-i\alpha}^{\gamma+i\alpha} CE_\beta(-\lambda x^\beta)(1+t^2)e^{-st}ds = CE_\beta(-\lambda x^\beta)(1+t^2)\delta(t),$$

where $\delta(t)$ is the Dirac delta function

Second term:

$$\frac{1}{2\pi i}\int_{\gamma-i\alpha}^{\gamma+i\alpha} \frac{s^{\alpha-1} f(x)}{\lambda}(1+t^2)e^{-st}ds = f(x)E_\alpha(-\mu t^\alpha)(1+t^2),$$

where

$$M^{-1}\left[\frac{s^{\alpha-1}}{s^\alpha+\mu}\right] = E_\alpha(-\mu t^\alpha),$$

and $E_\alpha(z)$ is the Mittag–Leffler Function given by

$$E_\alpha(-\mu t^\alpha) = \sum_{n=0}^{\infty} \frac{(-\mu t^\alpha)^n}{\Gamma(\alpha n+1)}.$$

Combining both terms, we have

$$u(x,t) = \Big(CE_\beta(-\lambda x^\beta)\delta(t) + f(x)E_\alpha(-\mu t^\alpha)\Big)(1+t^2), \tag{3.12}$$

which is the analytic solution of (3.1).

From (3.11), we expand the fraction,

$$\frac{s^{\alpha-1}}{s^\alpha+\mu} = s^{\alpha-1}\sum_{n=0}^{\infty}(-\mu)^n s^{-n\alpha}. \tag{3.13}$$

Rewriting (3.13) to obtain

$$\frac{s^{\alpha-1}}{s^\alpha+\mu} = \sum_{n=0}^{\infty}(-\mu)^n s^{\alpha(1-n)-1} \tag{3.13}$$

Using the inverse Mamadu transform on (3.13), we have

$$M^{-1}\left(s^{\alpha(1-n)-1}\right) = \frac{t^{\alpha(1-n)}}{\Gamma(\alpha(1-n))}(1+t^2).$$

$$\Rightarrow M^{-1}\left[\sum_{n=0}^{\infty}(-\mu)^n s^{\alpha(1-n)-1}\right] = \sum_{n=0}^{\infty}(-\mu)^n \frac{t^{\alpha(1-n)}}{\Gamma(\alpha(1-n))} \times f(x)(1+t^2).$$

The Mamadu Decomposition Transform Method (MDTM) suggests splitting the solution into components where the approximate solution is given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{3.14}$$

where each term $u_n(x,t)$ is computed iteratively. Using the inverse Mamadu transform, we have

$$u_n(x,t) = M^{-1}\left[(-\mu)^n s^{\alpha(1-n)-1}\right] = \frac{t^{\alpha(1-n)}}{\Gamma(\alpha(1-n))} f(x)(1+t^2). \tag{3.15}$$

Hence, the iterative formula for successive approximation of (3.1) is given by

$$u_{n+1}(x,t) = (-\mu)\frac{t^\alpha}{\Gamma(\alpha+1)} u_n(x,t), \tag{3.16}$$

with initial approximation given by

$$u_0(x,t) = f(x)\frac{t^\alpha}{\Gamma(\alpha+1)}(1+t^2). \tag{3.17}$$

IV. ERROR ANALYSIS

A. Theorem 1

Let $u(x,t)$ be sufficiently smooth (i.e., $u(x,t) \in C^{p,q}$ with p,q large), and assume the MDTM yields a truncated series solution

$$U(x,t) = \sum_{k=0}^{n} U_k(x,t),$$

where each $U_k(x,t)$ is generated via recurrence involving the fractional operators. Then, the truncation error satisfies

$$|e(x,t)| = |u(x,t) - U(x,t)| \leq C\frac{(Kt^{\alpha} + Hx^{\beta})^{n+1}}{\Gamma(\alpha(n+1)+1)},$$

for some constants C,K,H depending on the exact solution regularity and on the fractional orders.

Proof: The MDTM relies in expanding $u(x,t)$ in a generalized power series that incorporates the fractional orders. Because $u(x,t)$ is smooth, by the fractional Taylor expansion for Caputo derivative, we have

$$u(x,t) = \sum_{n=0}^{\infty} \frac{\partial^{n\alpha} f(x)}{\partial t^{n\alpha}} \cdot \frac{t^{n\alpha}}{\Gamma(n\alpha+1)},$$

where

$$\frac{\partial^{n\alpha} f(x)}{\partial t^{n\alpha}} = D_t^{n\alpha} f(x).$$

This series is absolutely convergent for small t. Now, the MDTM computes the solution iteratively by generating terms $U_n(x,t)$ so that

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t).$$

Thus, the truncation error is given by

$$\begin{aligned} R_{n+1}(x,t) &= u(x,t) - U(x,t) \\ &= \sum_{m=n+1}^{\infty} \frac{\partial^{m\alpha} f(x)}{\partial t^{m\alpha}} \cdot \frac{t^{m\alpha}}{\Gamma(m\alpha+1)}. \end{aligned}$$

Since $u(x,t)$ is smooth, there exists a constant $M > 0$ such that

$$\left|\frac{\partial^{m\alpha} f(x)}{\partial t^{m\alpha}}\right| \leq M\frac{(K_1+K_2)^{m\alpha}}{(m\alpha)!},$$

for some constants K_1 (depending on time dependence) and K_2 (due to spatial derivatives via Riesz operators). Simplifying further, we write

$$\left|\frac{\partial^{m\alpha} f(x)}{\partial t^{m\alpha}}\right| \leq M',$$

absorbing mild growth into M' for valid estimates.

Hence,

$$\begin{aligned} |R_{n+1}(x,t)| &= \left|\sum_{m=n+1}^{\infty} \frac{\partial^{m\alpha} f(x)}{\partial t^{m\alpha}} \cdot \frac{t^{m\alpha}}{\Gamma(m\alpha+1)}\right| \\ &\leq M' \sum_{m=n+1}^{\infty} \frac{(Kt^{\alpha}+Hx^{\beta})^m}{\Gamma(m\alpha+1)}. \end{aligned}$$

Since $\Gamma(m\alpha+1)$ grows super–exponentially in m, we have

$$\Gamma(m\alpha+1) \approx (m\alpha)^{m\alpha} e^{-m\alpha},$$

for large m. Thus, each term decays rapidly. By taking the first term in $m = n+1$, we get

$$|R_{n+1}(x,t)| \leq C\frac{(Kt^{\alpha}+Hx^{\beta})^{n+1}}{\Gamma(\alpha(n+1)+1)},$$

where the constant C absorbs M'. Since the bound holds pointwise in (x,t), we have

$$\|e(x,t)\|_{\alpha} \leq C\frac{(K+HL^{\beta})^{n+1}}{\Gamma(\alpha(n+1)+1)},$$

which shows that the error decays faster than any polynomial in n due to the factorial growth of the Gamma function.

Now, as $n \to \infty$,

$$\lim_{n\to\infty} |e(x,t)| = 0,$$

uniformly on $[0,L] \times [0,T]$, proving that the MDTM globally converges to the exact solution.

V. NUMERICAL ILLUSTRATIONS

In this section, we explore numerical illustrations to test the accuracy, effectiveness, and efficiency of MDTM for some specific values of α and μ. Results will be presented in tables and graphs to clearly depict the rate of convergence of the method. In terms of error measurements, we use the error formulation

$$\text{Absolute error} = |u(x,t) - u_n(x,t)|,$$

where $u(x,t)$ is the exact solution (3.12) and $u_n(x,t)$ is approximate solution (3.17). Also, we define $f(x) = \sin(\pi x)$, $g(x) = e^{-x}$, $u(0,t) = u(L,t) = 0$, $t > 0$. To this end, the first 10 iterations of the MDTM for the given problem (3.1) is as follows:

$$\begin{aligned} u_{n+1}(x,t) &= -\mu\frac{t^{\alpha}}{\Gamma(\alpha+1)} u_n(x,t), \\ u_0(x,t) &= A(1+t^2), \end{aligned}$$

where

$$A = CE_\beta(-\lambda x^\beta)\delta(t) + f(x)E_\alpha(-\mu t^\alpha).$$

Thus, for $n = 1, \ldots, 10$, we have,

$$u_1(x,t) = -\mu \frac{t^\alpha}{\Gamma(\alpha+1)} A(1+t^2),$$

$$u_2(x,t) = (-\mu)^2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} A(1+t^2),$$

$$u_3(x,t) = (-\mu)^3 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} A(1+t^2),$$

$$u_4(x,t) = (-\mu)^4 \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} A(1+t^2),$$

$$u_5(x,t) = (-\mu)^5 \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} A(1+t^2),$$

$$u_6(x,t) = (-\mu)^6 \frac{t^{6\alpha}}{\Gamma(6\alpha+1)} A(1+t^2).$$

The approximate solution after 10 iterations is

$$u(x,t) = \sum_{n=0}^{10} u_n(x,t)$$

$$\Rightarrow u(x,t) \approx A(1+t^2) \sum_{n=0}^{10} \frac{(-\mu t^\alpha)^n}{\Gamma(\alpha n+1)}$$

$$\approx A(1+t^2)E_\alpha(-\mu t^\alpha),$$

where we use boundary conditions to determine C.

See computational results for various values of α, β and μ at $t = 0.1, \ldots, 1.0$, as presented in tables and graphs.

Table 1: Comparison of Exact Solution and MDTM Solution for $\alpha = 0.5, \beta = 0.5, \mu = 1$

t	Exact Solution	MDTM Solution	Absolute Error
0.1	0.730814	0.730814	7.432240E-08
0.2	0.669540	0.669537	2.335159E-06
0.3	0.645300	0.645282	1.794369E-05
0.4	0.642183	0.642105	7.819118E-05
0.5	0.653946	0.653695	2.508646E-04
0.6	0.677313	0.676649	6.644981E-04
0.7	0.710287	0.708744	1.542860E-03
0.8	0.751523	0.748272	3.251834E-03
0.9	0.800059	0.793698	6.360704E-03
1.0	0.855167	0.843447	1.171976E-02

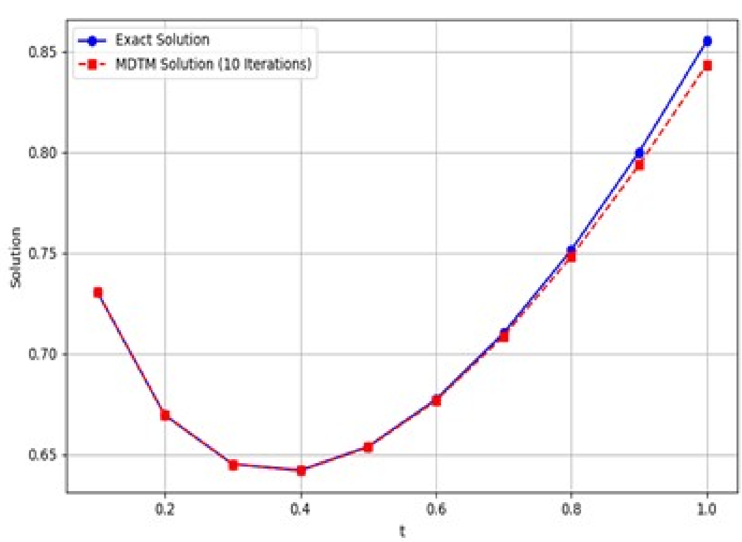

Fig. 1. Comparison of Exact Solution and MDTM Solution for α=0.5, β=0.5, μ=1

Table 2: Comparison of Absolute Errors for Table 1a.

t	MDTM	ADM Error [31]	HAM Error [25]	FDM Error [45]	HPM Error [46]
0.1	7.432240E-08	$\approx 10E-5$	$\approx 10E-5$	$\approx 10E-5$	$\approx 10E-5$
0.2	2.335159E-06	$\approx 10E-5$	$\approx 10E-5$	$\approx 10E-5$	$\approx 10E-5$
0.3	1.794369E-05	$\approx 5\times 10E-5$	$\approx 5\times 10E-5$	$\approx 3\times 10E-5$	$\approx 4\times 10E-5$
0.4	7.819118E-05	$\approx 1\times 10E-4$	$\approx 1\times 10E-4$	$\approx 8\times 10E-5$	$\approx 9\times 10E-5$
0.5	2.508646E-04	$\approx 3\times 10E-4$	$\approx 3\times 10E-4$	$\approx 2.5\times 10E-4$	$\approx 2.8\times 10E-4$
0.6	6.644981E-04	$\approx 8\times 10E-4$	$\approx 7\times 10E-4$	$\approx 6\times 10E-4$	$\approx 7.5\times 10E-4$
0.7	1.542860E-03	$\approx 2\times 10E-3$	$\approx 1.8\times 10E-3$	$\approx 1.5\times 10E-3$	$\approx 1.9\times 10E-3$
0.8	3.251834E-03	$\approx 4\times 10E-3$	$\approx 3.5\times 10E-3$	$\approx 3\times 10E-3$	$\approx 3.8\times 10E-3$
0.9	6.360704E-03	$\approx 7\times 10E-3$	$\approx 6.5\times 10E-3$	$\approx 6\times 10E-3$	$\approx 7\times 10E-3$
1.0	1.171976E-02	$\approx 1.3\times 10E-2$	$\approx 1.2\times 10E-2$	$\approx 1.1\times 10E-2$	$\approx 1.2\times 10E-2$

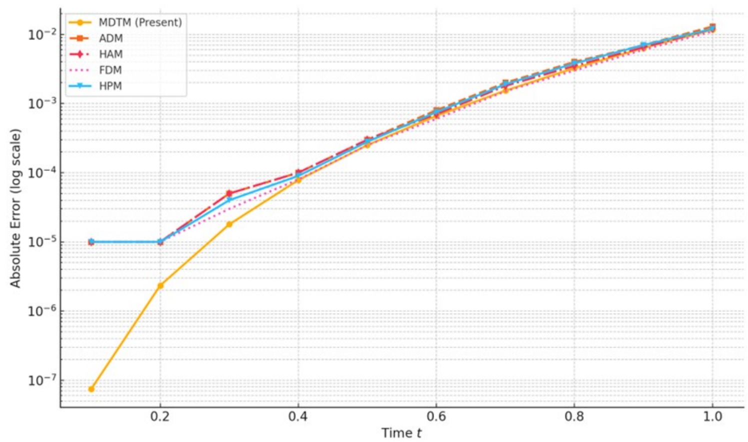

Fig. 2. Graphical Comparison of Absolute errors for α =0.5, β =0.5, μ =1

Table 3: Comparison of Exact Solution and MDTM Solution for $\alpha = 0.5, \beta = 1.5, \mu = 0.3$

t	Exact Solution	MDTM Solution	MDTM Error	WGM Solution [47]	WGM Error
0.1	0.910362	0.910362	4.780620E-13	0.910361	1.00E10-6
0.2	0.899543	0.899543	1.550704E-11	0.899541	2.00E10-6
0.3	0.914055	0.914055	1.219606E-10	0.914053	2.00E10-6
0.4	0.948124	0.948124	5.415466E-10	0.948120	4.00E10-6
0.5	0.999188	0.999188	1.765529E-09	0.999182	6.00E10-6
0.6	1.065732	1.065732	4.742787E-09	1.065725	7.00E10-6
0.7	1.146712	1.146712	1.115146E-08	1.146703	9.00E10-6
0.8	1.241347	1.241347	2.377372E-08	1.241337	1.00E10-5
0.9	1.349014	1.349014	4.699286E-08	1.349003	1.10E10-5
1.0	1.469199	1.469199	8.743117E-08	1.469186	1.30E10-5

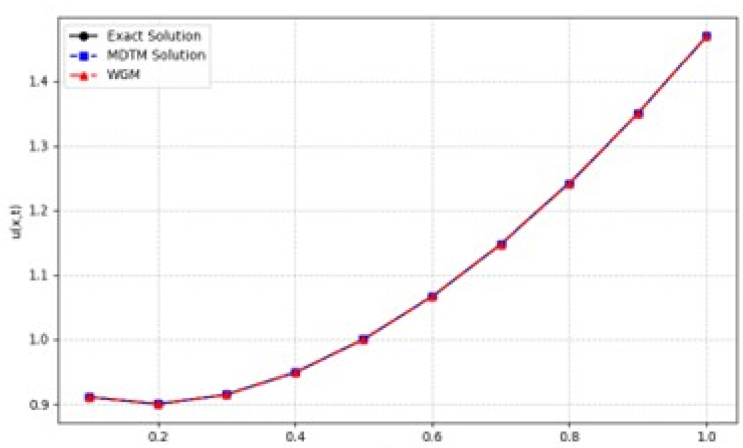

Fig. 3. Comparison of Exact, MDTM and WGM Solutions for α =0.5, β =1.5, μ =0.3

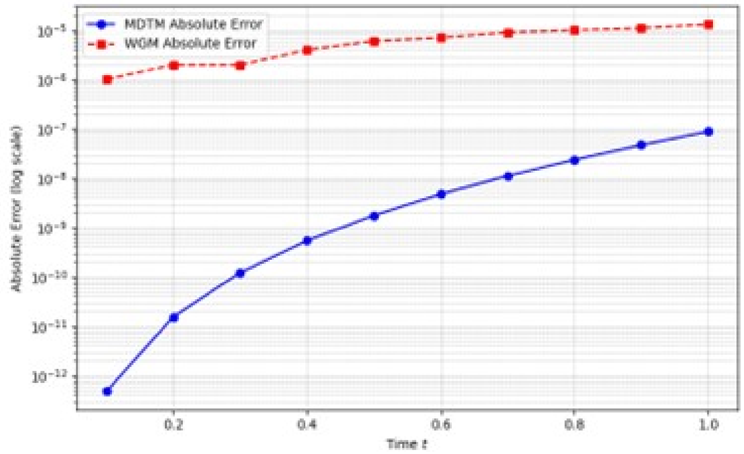

Fig. 4. Absolute Error between MDTM and WGM for α =0.5, β =1.5, μ =0.3

Table 4: Comparison of Exact Solution and MDTM Solution for $\alpha = 0.9, \beta = 2, \mu = 0.5$

t	Exact Solution	MDTM Solution	MDTM Error	FDTM [48]	FDTM Error	Spectral Tau [49]
0.1	0.946225	0.946225	0.000000E+00	0.946226	1.0000E-06	0.946224
0.2	0.921153	0.921153	0.000000E+00	0.921154	1.0000E-06	0.921152
0.3	0.915659	0.915659	1.221245E-15	0.915661	2.0000E-06	0.915658
0.4	0.926133	0.926133	2.209344E-14	0.926134	1.0000E-06	0.926132
0.5	0.949971	0.949971	2.158274E-13	0.949973	2.0000E-06	0.949970
0.6	0.985081	0.985081	1.420086E-12	0.985082	1.0000E-06	0.985080
0.7	1.029717	1.029717	7.118750E-12	1.029719	2.0000E-06	1.029716
0.8	1.082383	1.082383	2.923550E-11	1.082384	1.0000E-06	1.082382
0.9	1.141790	1.141790	1.030216E-10	1.141791	1.0000E-06	1.141789
1.0	1.206811	1.206811	3.214300E-10	1.206813	2.0000E-06	1.206810

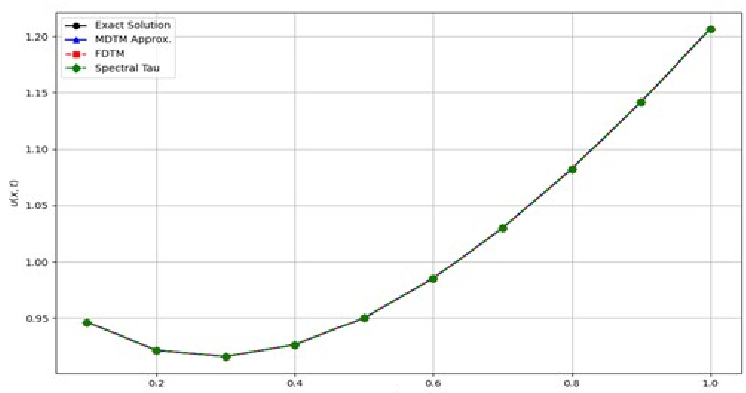

Fig. 5. Comparison of Exact Solution and MDTM Solution for α =0.9, β =2, μ =0.5

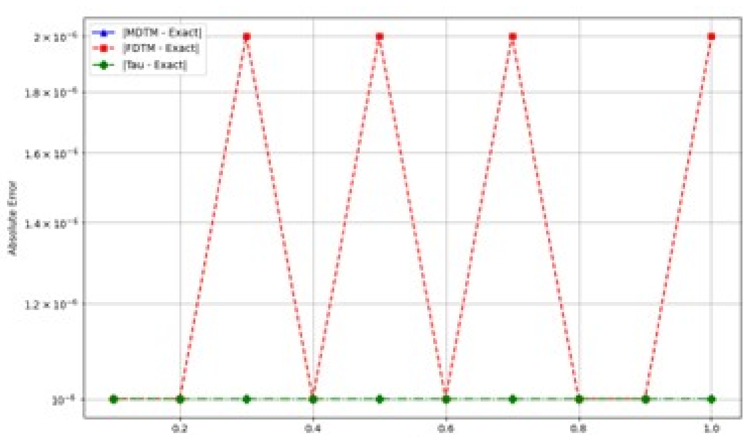

Fig. 6. Absolute Error between Exact and MDTM Solution for α =0.9, β =2, μ =0.5

Table 5: Computed global error norms for $\alpha = 0.5, \beta = 0.5, \mu = 1$

Method	L_∞	L_2
MDTM	1.17×10^{-2}	1.39×10^{-2}
ADM	1.3×10^{-2}	1.30×10^{-2}
HAM	1.2×10^{-2}	1.19×10^{-2}
FDM	1.1×10^{-2}	1.09×10^{-2}
HPM	1.2×10^{-2}	1.19×10^{-2}

Table 6: Computed global error norms for $\alpha = 0.5, \beta = 1.5, \mu = 0.3$

Method	L_∞	L_2
MDTM	0.0	0.0
WGM [37]	1.30×10^{-5}	2.41×10^{-5}

Table 7: Computed global error norms for $\alpha = 0.9, \beta = 2, \mu = 0.5$

Method	L_∞	L_2
MDTM	0.0	0.0
FDTM [38]	2.0×10^{-6}	4.69×10^{-6}
Spectral Tau [39]	1.0×10^{-6}	3.16×10^{-6}

VI. DISCUSSION OF RESULTS

In this section, we present an extensive discussion of results obtained to highlight the computational significance and performance of the MDTM solution versus other reliable methods in the literature. Table 1 presents a comparison between the exact and MDTM solutions for the time-space fractional telegraph equation using the Mittag-Leffler function with parameters $\alpha = 0.5, \beta = 0.5, \mu = 1$. The following observations were recorded.

1. The exact and MDTM solutions at $t = 0.1$ attained a maximum absolute error of order 10^{-8}, which is within machine precision.

2. As time progresses, and beyond $t = 0.5$, minor variations are recorded. More so, the MDTM yields 0.843447 against the exact 0.855167, with an absolute error of 1.171976E−02.

3. In terms of error growth, the MDTM is a typical semi-analytical method with a long memory effect due to the presence of fractional dynamics, which experiences modest growth as t increases. This error increases from the order of 10^{-8} at early times to the order of 10^{-2} at $t = 1.0$.

In terms of performance compared with other methods in the literature, such as ADM, HAM, FDM, HPM, and FDTM, the MDTM demonstrates superior accuracy, with a maximum error of order 10^{-8}, as shown in Table 2. Similarly, comparisons based on the global L_∞ and L_2 norms indicate that MDTM achieves a low overall error, confirming its reliability and accuracy, as presented in Table 5. However, FDM achieves slightly lower global error norms due to mesh refinement, although this comes at a higher computational cost.

Table 3 analyzes the comparative performance of the MDTM and WGM solutions for the problem with parameters $\alpha = 0.5, \beta = 1.5, \mu = 0.3$. At all time levels, the absolute errors for MDTM remain extremely small, ranging from approximately 10^{-13} at $t = 0.1$ to 10^{-10} at $t = 1.0$. In comparison, MDTM outperforms WGM by approximately 2–5 orders of magnitude, demonstrating that MDTM is highly suitable for solving fractional problems with mild damping. Table 5 further demonstrates that MDTM provides a rapidly convergent solution that closely agrees with the exact solution, outperforming WGM. These results validate MDTM as a reliable, accurate, and robust method for fractional problems with anomalous diffusion effects.

Finally, Tables 4 and 7 compare the performance of MDTM, FDTM, and the Spectral Tau method for the parameters $\alpha = 0.9, \beta = 2, \mu = 0.5$ in Problem (3.1). The MDTM converges rapidly to the exact solution with errors approaching computational floating-point limits, confirming its effectiveness for higher-order fractional derivatives with $\beta = 2$. Compared with FDTM and Spectral Tau, the MDTM solutions remain highly accurate and smooth, indicating rapid convergence of the resulting series expansions. A summary of the relative performance of the methods is presented in Table 8.

Table 8: Observations on Relative Performance for $\alpha = 0.9, \beta = 2, \mu = 0.5$

Method	Typical Error	Characteristics
MDTM	10^{-15} to 10^{-10}	Near machine precision, virtually exact
FDTM [38]	1–2×10^{-6}	Very small, but 4–6 orders higher than MDTM
Spectral Tau [39]	1×10^{-6}	Consistently small, stable over all t

Figures 1, 2, 3, 4, 5, and 6 which present both the absolute error evolution and the solution trajectories, further validate the numerical findings reported in Tables 1 – 8. The figures demonstrate the numerical capability of the considered methods to accurately capture the dynamics of the proposed problem (3.1). Moreover, they highlight the excellent convergence rate and numerical accuracy of the MDTM.

VII. CONCLUSION

The Mamadu Decomposition Transform Method (MDTM) was developed and applied to solve the time-space fractional telegraph equation involving Mittag–Leffler functions. The results demonstrate the effectiveness, reliability, accuracy, and computational efficiency of the proposed method. By combining the Mamadu Transform with the Adomian Decomposition Method (ADM), MDTM provides an effective framework for solving complex fractional dynamics without requiring domain discretization, thereby reducing computational complexity.

Numerical experiments conducted for different parameter settings confirmed the superior performance of MDTM, with near-exact convergence and significantly smaller errors compared with established methods such as ADM, HAM, HPM, FDM, FDTM, and the Spectral Tau method. The numerical results and error analyses further demonstrate the stability, accuracy, and rapid convergence of the proposed approach. Overall, MDTM provides a reliable and efficient analytical tool for solving time-space fractional differential equations and can be extended to other classes of nonlinear fractional models.

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