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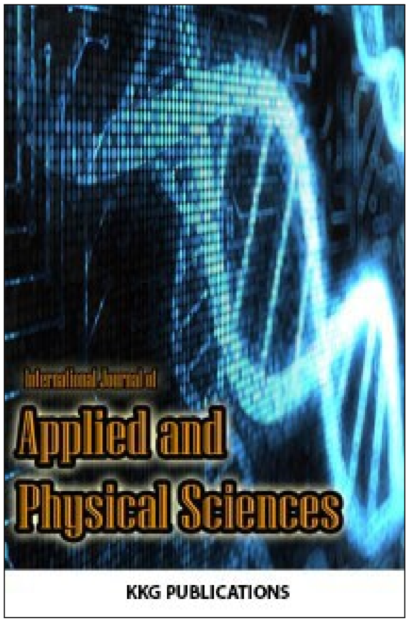

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Published online: 14 July 2017

To cite this article: M. Erfanian and A. Akrami, "Using of two dimensional HAAR wavelet for solving of two dimensional nonlinear fredholm integral equation," *International Journal of Applied and Physical Sciences,* vol. 3, no. 2, pp. 55-60, 2017.
DOI: https://dx.doi.org/10.20469/ijaps.3.50006-2

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International Journal of Applied and Physical Sciences
2017, 3(2): 55-60

IJAPS

USING OF TWO DIMENSIONAL HAAR WAVELET FOR SOLVING OF TWO DIMENSIONAL NONLINEAR FREDHOLM INTEGRAL EQUATION

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Keywords:
Two-Dimensional Nonlinear Fredholm Integral Equations
2D RH Wavelet
Banach Fixed Point Theorem

Received: 09 March 2017
Accepted: 14 May 2017
Published: 14 July 2017

Abstract. In this work, we used the properties of the two-dimensional Haar wavelet; for this purpose, it is required to define the integral operator and obtain an operational matrix for our integral equation. Also, we used from 2D-Haar wavelet to approximate solutions of nonlinear two-dimensional Fredholm integral equations without solving a linear system. In section error analysis, we apply Banach fixed point theorem, and we proved my integral operator has a unique fixed point. In section four, numerical example, we choose one example, and we have compared my method with the other methods. It has been observed that the approximation solutions are obtained are very suitable. Moreover, the CPU runs times in seconds are presented. Also, we can expand this method to another type of 2D integral equation, such as Volterra or mixed of Volterra and Fredholm.

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INTRODUCTION

Many types of problems in engineering science such as telegraph equations [1], electrical engineering [2], electrochemical process, heat and mass transfer [3] electromagnetic and electrodynamic [4] molecular physics, fluid mechanic, Consolidation equation, acoustic, chemical [5], seepage diffusion equation, population [6], can be change to 2D integral equation. A lot of work has been done in field of two-dimensional integral equations, for instance, G. Han, in [7], [8] was Solved 2D Fredholm integral equations by the Galerkin iterative method and linear Elements, for the first time. In [9] with using Gaussian radial basis function, and in [10] solving by triangular orthogonal functions (2D-TFs). Also in [11] using of HAAR wavelet. In [12] method and in [13] from meshless method solving this equation. Legendre polynomial and interpolation methods [14], [15], differential transform and Gauss product quadrature rule method, hat functions and with Piecewise Intervals [16, 17, 18, 19, 20] with Chebyshev hybrid functions solved two-dimensional integral equations and in [21] Graham, applied Collocation method for two-dimensional weakly singular integral equations. Also, there exist some work for approximate of two-dimensional Volterra integral equations such as in [22, 23, 24] for 2D nonlinear Volterra, and in [25] for linear Volterra integral, furthermore, in [26] using of Bernstein polynomials method for solving two-dimensional Volterra-Fredholm integral equations, also in [27] solved 2D integral equations of the first kind by multi-step methods. In [28], with Haar wavelet obtained Numerical solutions of nonlinear two-dimensional partial Volterra integro-differential equations.

LITERATURE REVIEW

In this work, we using of 2D Rational Haar wavelet method for solving 2D Fredholm integral equation. Haar wavelets one of the simplest wavelets among various types of wavelets, and the RH functions are family on [0, 1] of only three values +1 , -1 and 0. That Alfred Haar in his PhD thesis in 1910 was presented of first example of a Haar function. In 1981 Jean Morlet, explanation of concerning wavelets and in 1984 Alex Grossman invented the term wavelet. Yves Meyer introduce Meyer wavelet in 1985 [29], [30]. Daubechies in [31] applied wavelet in mathematics, signal processing, and numerical analysis in 1988. A lot of types of wavelets are existed such as Haar, Legendre, Legendre multiwavelets, Chebyshev, Coiflet, Mathieu, Poisson, Shannon, Spline, and Stromberg wavelet [32]. We applied Haar wavelet in one-dimensional integral equation, such as in [33] for nonlinear Fredholm integral equations, in [34] for nonlinear mixed Volterra Fredholm integral equations, in [35] we extended for numerical solution of integro-differential equation, and in [36] for nonlinear Volterra Fredholm Hammerstein integral equations.

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METHODOLOGY

Consider the 2D Fredholm integral equation of below:

$$v(x,y) = g(x,y) + \int_0^1 \int_0^1 U(x,y,t,s,v(s,t))dsdt. \quad (1)$$

That $v \in C([0,1]^2)$ is an unknown function, $f : [0,1]^2 \to R^2$, and $U : [0,1]^4 \times R^2 \to R^2$, are a known continuous function, and for Lipschitz function of U, we have
$|U(x,y,t,s,v_1(s,t)) - U(x,y,t,s,v_2(s,t))| \leq L|v_1 - v_2|$,
where $L \geq 0, v_1, v_2 \in R^2$.

One of the most efficient and effective methods for solving the equation (1) is using the 2D Haar wavelets basis. We applied 2D-Haar wavelet to find numerical of solutions. To achieve this purpose, we define operator T in the Banach space that are continuous and real valued functions
$T : (C([0,1]^2), ||.||_\infty) \to (C([0,1]^2), ||.||_\infty)$,
with using this operator, we have

$$T(v(x,y)) = g(x,y) + \int_0^1 \int_0^1 U(x,y,t,s,v(s,t))dsdt. \quad (2)$$

Therefore, from assumptions in [37] and the Banach fixed point theorem in section Error Analysis we proved T, has a unique fixed point; thus, the integral equation (1) has exactly one solution.

Rational Haar Functions

Definition 2, 1

The function of RH wavelet on [0,1] is defined as follows:

$$H(t) = \begin{cases} 1 & 0<t \leq 1/2, \\ -1 & 1/2<t<1, \\ 0 & \text{otherwise} \end{cases}$$

and for all $t \in [0,1)$
$h_0(t) = 1$.

Also, we can be rewritten the RH function by
$h_n(t) = H(2^p t - q), n = 2^p + q$
with $p = 0, 1, \ldots$ and $q = 0, 1, \ldots, 2^p - 1$.
We define $h(t) = [h_0(t), h_1(t), \ldots, h_(m-1)(t)]^T$, and
$\hat{\phi}_{m\times m} = \left[h(\frac{1}{2m}), h(\frac{3}{2m}), ..., h(\frac{2m-1}{2m})\right]$.
That $m = 2^(n+1) \in N$, for example, the first eight RH functions can be written in the matrix form as

$$\hat{\phi}_{8\times 8} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Also for any function two variable we can be expanded by 2D Haar wavelets functions then we have

$$v(s,t) = \sum_{i=0}^{m} \sum_{j=0}^{m} a_{ij} h_{ij}(s,t) = F^T M(s,t). \quad (3)$$

In this equation
$h_{ij}(s,t) = h_i(s)h_j(t)$,
and

$$a_{ij} = \langle h_i(s), \langle v(s,t), h_j)(t) \rangle \rangle. \quad (4)$$

We can show vectors F and M are below

$$F = [a_{0,0}, a_0, 1, ..., a_{0,(m-1)}, a_1, 0, ..., a_{1,(m-1)}, ..., a_{(m-1),0}, ..., a_{(m-1),(m-1))}] \quad (5)$$

and

$$M(s,t) = [h_{00}, ..., h_{o(m-1)}, h_{10}, ..., h_{1(m-1)}, ..., h_{(m-1)0}, ..., h_{(m-1)(m-1)}]^T(s,t), \quad (6)$$

we also have:
$\int_0^1 \int_0^1 h_{m,n}(x,t)h_{l,q})(x,t)dxdt$
$= \int_0^1 h_m(x)h_l(x)dx \int_0^1 h_n(t)h_q(t)dt$
$= \begin{cases} 1, & l = m, n = q, \\ 0 & \text{otherwise} \end{cases}$
We assume in equation of (1)

$$U(x,y,t,s,v(s,t)) = \sum_{(i=1)}^{l} U_i(x,y)V_i(s,t), \quad (7)$$

and functions of, V_i (s,t), will be approximated with Haar wavelets, so we have

$$\phi_i(s,t)V_i(s,t) = M^T(s,t)A_i M(s,t), \quad (8)$$

we assume Q_m is an orthogonal projection with interpolation condition, and

$$\phi_{n-1}(s,t) = U(x,y,t,s,v_{n-1}(s,t)). \quad (9)$$

Then by using of RH functions for $\psi_{n-1}(s,t)$ we have

$$Q_m(\phi_{n-1}(s,t)) = \sum_{l=1}^{r} U_l(x,y)M^T(s,t)A_l M(s,t), \quad (10)$$

that

$$A_l = [a_{pq}^{(l)}]_{m\times m'} l = 1, 2, 3, ..., r, \quad (11)$$

and $m = 2^{n+1} \in N$, that $n \geq 1$

$$a_{pq}^{(l)} = 2^{\frac{i+j}{2}} \langle h_p(s), \langle \phi_t(s,t), h_q(t) \rangle \rangle, \quad (12)$$

with i,j = 0,1,..., where
$p = 2^j + k \qquad k = 0, 1, ..., 2^j - 1$,

$q = 2^i + k' \qquad k' = 0, 1, ..., 2^i - 1.$

Thus by using this equations for l=1,2,3,...,r we have

$$A_l = (\hat{\phi}^{-1}_{m\times m})^T . \hat{A}_l (\hat{\phi})^{-1}_{m\times m'} \tag{13}$$

where

$$\hat{A}_l = [(\hat{a}^{(l)})_{pq}]_{m\times m'} p, q = 1, 2, ..., m, \tag{14}$$

as

$$(\hat{a}^l)_{pq} = \phi(\frac{2p-1}{2m}, \frac{2q-1}{2m}). \tag{15}$$

Thus for the 2D Fredholm integral equation as we have

$$v_i(x, y) = g(x, y) + \int_0^1 \int_0^1 Q_m(\psi_i(s, t))dtds, i = 1, 2, 3...$$
(16)

Error Analysis

Since we using of 2D Haar wavelets for approximated of 2D nonlinear Fredholm integral equation, so with the help of the following theorem convergence and upper bound of the equation (1) is evaluated.

Lemma 3-1

Let Lipschitz function of U from $[0, 1]^4 \times R^2 \rightarrow R^2$, and $|U(x, y, t, s, v_1(s, t)) - (x, y, t, s, v_2(s, t))| \leq L|v_1 - v_2|$,

that L is a Lipschitz constant, then operator T in equation of (2) has an unique fixed point of v and for all initial point of v_0 we have

$$\| v - T^i(v_0) \|_\infty \leq \| T(v_0) - v_0 \|_\infty \times \sum_{(j=i)}^{\infty} L^j, \tag{17}$$

that L <1, and

$$\| . \|_\infty = sup\{|g(s, t)|; (s, t) \in [0, 1]\times[0, 1], g : [0, 1]^2 \rightarrow R^2\},$$
(18)

Proof:

If v_1, v_2, are two unknown function in $C([0, 1]^2)$, for the 2D Fredholm integral equations then we have

$|T(v_1(x, y)) - T(v_2(x, y))| =$

$| \int_0^1 \int_0^1 U(x, y, t, s, v_1(s, t))$

$-U(x, y, t, s, v_2(s, t))dtds|$

$\leq \int_0^1 \int_0^1 |U(x, y, t, s, v_1(s, t))$

$-U(x, y, t, s, v_2(s, t))dtds|$

$\leq L \int_0^1 \int_0^1 |v_1(s, t) - v_2(s, t)|dtds \leq L||v_1 - v_2||_\infty.$

By induction, for the 2D Fredholm integral equation and every $n \in N$ we have

$||T^n(v_1) - T^n(v_2)||_\infty \leq L^n||v_1 - v_2||_\infty,$

Therefore, the equation of (2) has a unique answer and (17) verify to the Banach fixed-point theorem.

Theorem 3-1

If we let that ϕ belong to$C([0, 1]^2)$ and $\{v_i \subset C([0, 1]^2), i = 1, 2, ...\}$, thus for the Lipschitzian function U we have

$$\| v - v_i \|_\infty \leq \| T(v_0) - v_0 \|_\infty \sum_{j=1}^{\infty} L^j + \sum_{j=1}^{i} L_{i-j_{\in_j}}. \tag{19}$$

Proof: If

$L_{i-j} = max\{\| \frac{\partial\phi_{i-1}}{\partial t} \|_\infty, \| \frac{\partial\phi_{i-1}}{\partial s} \|_\infty\},$

that i=0,1,. . . , and $m = 2^(i + 1)$ thus for equation (1) we have

$\| T(v_{i-1}) - vi \|_\infty \left\| \int_0^1 \int_0^1 (\phi_{i-1(s,t)} \right.$

$\left. - Qm(\phi_{i-1}(s, t))dtds \right\|$

$\leq \left\| \phi_{i-1} - Q_m(\phi_{i-1}) \right\| \infty,$

So if we set

$g(s, t) := \phi_{i-1} - Q_m(\phi_{i-1}),$

And using of the mean value theorem and interpolating property we have

$t_i = \frac{1}{2^{d_i=1}} + \frac{v_1}{2^d_1}, s_j = \frac{1}{2^d_2+1} + \frac{v_2}{2^d_2},$

Where $t_0 = 0$, and

$i = 2^d_1 + v_1, \qquad d_1, d_2 \geq 1,$

$j = 2^d_2 + v_2, \qquad i, j \leq m - 1,$

we have

$\| \phi_{i-1} - Q_m(\phi_{i-1}) \| \infty$

$\| g(s_j, t_j) + \frac{\partial_g}{\partial_t}(\xi, \gamma)(\xi - t_j)$

$\frac{\partial_g}{\partial_t}(\xi, \gamma)(\gamma - s_j) \| \infty$

$=\| (1 - Q_m)\frac{\partial\phi_{i-1}}{\partial_t}(\xi, \gamma)(1 - Q_m)\frac{\partial\phi_{i-1}}{\partial_s}(\xi, \gamma) \| \infty$

$\max\{\| \xi - t_i \| \infty, \| \gamma - s_j \| \infty\}$

$\leq \frac{2}{2^i} \| (1 - Q_m \|)\infty\|\frac{\partial\phi_{i-1}}{\partial_t}(\xi, \gamma) + \frac{\partial\phi_{i-1}}{\partial_s}(\xi, \gamma) \| \infty$

$\leq \frac{4L_{i-1}}{2^1},$

thus we have

$\| T(v_{i-1}) - v_i \| \infty \leq \frac{4L_{i-1}}{2^i}$

If

$\frac{4L_{k-1}}{2^k} < \varepsilon_k, \qquad k = 1, 2, \ldots, i,$

that $\varepsilon_1, \varepsilon_2, , , , , \varepsilon_3 > 0$ for $i \geq 1$, we have

$$\| T(v_{i-1}) - v_i \| \infty < \varepsilon_i \tag{20}$$

By using of the triangle inequality we have

$$\| (v - v_i) \| \infty \leq |u - T^i(v_0)|\infty + \sum_{j=1}^{i} L_{i-j} \| Tv_{j-1} - v_j \| \infty$$
(21)

so, from (20) and (21) we conclude

$$\| v - v_i \| \infty \leq \| T(v_0) - v_0 \| \infty \sum_{j=i}^{\infty} L_j + \sum_{j=i}^{i} L_{i-j}\varepsilon_j$$

Numerical Examples

We have applied the method described in equation of (16) for solving some example from various references. Also, this method some advantage for example, we dont need to solving any nonlinear numerical equation system, in this method CPU time is very low and the solving of equations with this method is an economical, as well, we define the sequence of approximating functions v_i for i=1,2,..., with an initial function $v_0 \varepsilon C([0,1]^2)$. In this section points are proposed is a $(x_i, t_i) = \left(\frac{1}{2^i}, \frac{1}{2^i}\right)$, for i=1,2,...,6. In addition, we can also mention the following algorithm, for this method.

Algorithm

1. produce matrices M(s,t), and$\hat{\phi}^{-1}_{m \times m}$.
2. For i=1 to k do,
3. Product Matrix $A_l, \hat{A}_1$ from (13) and (14).
4. Compute $Q_m(\phi_(i-1)(s,t))$from (11).
5. Compute v_i (x,y) from (16) for the supposed point.
6. Go to step 2.

Example 5.1 let the 2D nonlinear Fredholm integral equation of below from [38]

$u(x,y) = f(x,y) + \int_0^1 \int_0^1 = f(x,y)\{t.sin(s) + 1)u(t,s)\}dtds,$

that

$f(x,y) = x.cos(y) - \frac{1}{6}sin(1)(3 + sin(1)),$

and the exact solution is $u(x,y) = x.cos(y)$.

In table1 we compared the maximum absolute errors of our method and Haar wavelet methods [6, 27]. Also we have shown an absolute error of Example 5.1 for m=128, in fig 1. CPU time for this example for m=128 is about 79,062 seconds.

TABLE 1
COMPARISON BETWEEN THE MAXIMUM ABSOLUTE ERRORS OF OUR METHOD AND HAAR WAVELET METHODS

i	m	Haar Wavelet Method ([11])	Haar Wavelet Method ([39])	Presented Method
1	2	8.6×10^{-3}		8.90×10^{-3}
2	4	1.2×10^{-2}	5.2×10^{-2}	1.14×10^{-4}
3	16	8.9×10^{-3}	2.1×10^{-2}	2.19×10^{-4}
4	32	2.0×10^{-2}	6.8×10^{-3}	3.40×10^{-5}
5	64	6.0×10^{-3}	1.9×10^{-3}	5.24×10^{-5}
6	128	4.3×10^{-5}		8.14×10^{-6}

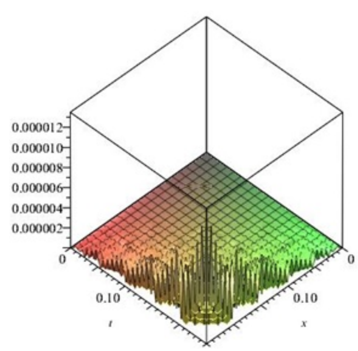

Fig. 1 . Example of absolute error

CONCLUSION

One of the most efficient and effective methods for solving the equation (1) is, using the Haar wavelets basis. Our work was discussed on using some properties of the 2D Haar wavelet basis. In this paper, we introduce an approximate method for solving of 2D nonlinear Fredholm integral equations, we applied 2D-Haar wavelet to find numerical of solutions, we will try to get an upper bounded for equations discussed and with using of Banach fixed point theorem we proved the convergence theorem of our method. In section, numerical example we choose one example from [38] and we compared the maximum absolute errors of our method with Haar wavelet methods [11], [39]. It has been observed that the approximation solutions are obtained are very suitable. Moreover, the CPU runs times in seconds are presented. Also, we can expand this method to another type of 2D integral equation, such as Volterra or mixed of Volterra and Fredholm.

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— This article does not have any appendix. —